

DYNAMICAL SUPERSYMMETRY OF A VORTEX SYSTEM WITH $1/r^2$ POTENTIAL

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Abstract

The Pauli Hamiltonian for a spin $\frac{1}{2}$ charged particle interacting with a point magnetic vortex and $1/r^2$ potential exhibits a dynamical supersymmetry $Osp(1,1)$ on the plane except at the origin. Using this symmetry, the spectrum and the wave functions have been obtained. And, the dynamical supersymmetry could be imported to the case when an external harmonic potential is added.

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I. INTRODUCTION

Several quantum mechanical systems can be completely and explicitly solved with the aid of their dynamical symmetries. The quantum mechanical system of a charged particle interacting with a Dirac magnetic monopole possesses a dynamical $O(2,1)$ symmetry [1]. The Pauli Hamiltonian for a spin $\frac{1}{2}$ charged particle in the presence of a magnetic monopole exhibits dynamical supersymmetry $Osp(1,1)$ [2,3]. Adding $1/r^2$ potential to the Pauli Hamiltonian, the dynamical symmetry is enlarged to $Osp(2,1)$ [3]. The spectrum and wave functions of such systems can be constructed by means of the group theoretical methods.

Recently, It has been shown that the quantum mechanical system of a charged particle interacting with a point magnetic vortex can be solved using the dynamical symmetry $O(2,1)$ [4–6]. Interactions of charged particles with point magnetic vortices in 2+1 dimensions frequently appear in physically interesting systems. A typical example is a system of a charged particle interacting with a long and thin solenoid. We can describe such a system with the Hamiltonian for a charged particle interacting with a point magnetic vortex in 2+1 dimensions. The Pauli Hamiltonian of a spin $\frac{1}{2}$ charged particle in the presence of a magnetic vortex has dynamical supersymmetry $sp^*(2,1)$ [5]. Then, we can ask what the dynamical supersymmetry of a spin $\frac{1}{2}$ charged particle interacting with a point magnetic vortex and $1/r^2$ potential is. We are going to realize the dynamical supersymmetry explicitly and to find the spectrum and wave functions of this system using the group theoretical methods.

II. DYNAMICAL SUPERSYMMETRY OF THE SYSTEM

In this section, we show that the Pauli Hamiltonian of a charged spin $\frac{1}{2}$ particle interacting with a point magnetic vortex and $1/r^2$ potential has dynamical supersymmetry $Osp(1,1)$. The Pauli Hamiltonian in 2+1 dimensions which includes $1/r^2$ potential is given by

$$H = \frac{1}{2m}(p_i - eA_i)^2 - \frac{e}{m}B\sigma_3 + \frac{\lambda^2}{2mr^2} + \frac{\lambda}{2mr^3}\sigma_i r_i, \quad i = \{1, 2\}, \quad (1)$$

where m , e and λ are the particle's mass, charge and a real parameter respectively. The vector potential A_i and the corresponding magnetic field B concentrated at the origin are externally given by

$$A_i(\mathbf{r}) = \frac{\Phi}{2\pi}\epsilon^{ij}\frac{\hat{r}^j}{r}, \quad (2)$$

$$B = \nabla \times \mathbf{A} = \Phi\delta^{(2)}(\mathbf{r}), \quad (3)$$

where Φ is the total magnetic flux. The Hamiltonian H can be derived from the Lagrangian

$$L = \frac{1}{2}m\dot{r}_i^2 + eA_i\dot{r}_i - \frac{\lambda^2}{2mr^2} + \frac{i}{2}\psi_\mu\dot{\psi}_\mu + \frac{e}{m}B\mathcal{S} + \frac{i\lambda}{mr^3}\psi_i\psi_0r_i, \quad \mu = \{0, 1, 2\}, \quad (4)$$

where \mathcal{S} is the spin, $\mathcal{S} = -\frac{i}{2}\epsilon_{ij}\psi_i\psi_j$. $\{\psi_\mu\}$ are real generators of the Grassmann algebra, which describe the spin degrees of freedom of a classical spin $\frac{1}{2}$ particle [7]. The rotationally invariant Lagrangian (4) gives the following total angular momentum J :

$$J = \mathbf{r} \times m\mathbf{v} + S + \nu, \quad (5)$$

where $\nu = \frac{e\Phi}{2\pi}$.

It is well known that if a Lagrangian is changed by an exact differential under an infinitesimal transformation, then there exists a corresponding Noether charge. Consider an infinitesimal transformation of the dynamical variables r_i and ψ_μ

$$r_i \rightarrow r_i + \delta r_i, \quad \psi_\mu \rightarrow \psi_\mu + \delta\psi_\mu. \quad (6)$$

If a Lagrangian is changed by a total time-derivative

$$\delta L = \frac{d}{dt}\Lambda \quad (7)$$

under the transformation (6), the conserved Noether charge \hat{O} is

$$\hat{O} = \delta r_i \frac{\partial L}{\partial \dot{r}_i} + \delta\psi_\mu \frac{\partial L}{\partial \dot{\psi}_\mu} - \Lambda. \quad (8)$$

The change of Lagrangian (4) under the transformation (6) is

$$\delta L = \delta L_1 + \delta L_2 + \delta L_3 , \quad (9)$$

$$\delta L_1 = \frac{1}{2}m\dot{r}_i\delta\dot{r}_i + \frac{i}{2}\delta\psi_\mu\dot{\psi}_\mu - \frac{i}{2}\dot{\psi}_\mu\psi_\mu , \quad (10)$$

$$\delta L_2 = eA_i\delta r_i + e\partial_i A_j\dot{r}_j\delta r_i + \frac{e}{m}\partial_i B\delta r_i\mathcal{S} - i\frac{e}{m}B\delta\psi_i\epsilon_{ij}\psi_j , \quad (11)$$

$$\delta L_3 = \frac{\lambda^2 r_i}{mr^4}\delta r_i + \frac{i\lambda}{m}\partial_i\left(\frac{r_j}{r^3}\right)\delta r_i\psi_j\psi_0 + \frac{i\lambda r_i}{mr^3}\delta\psi_i\psi_0 - \frac{i\lambda r_i}{mr^3}\delta\psi_0\psi_i . \quad (12)$$

Therefore, under the super-transformations

$$\delta r_i = i\alpha f(t)\psi_i , \quad \delta\psi_i = \alpha m(-f(t)\dot{r}_i + \dot{f}(t)r_i) , \quad \delta\psi_0 = \alpha f(t)\frac{\lambda}{r} , \quad (13)$$

where $f(t) = \{1, t\}$ and α is a Grassmann super-transformation parameter, the Lagrangian (4) is changed by a total time-derivative

$$\delta L = \frac{d}{dt}\left(i\alpha f e A_i\psi_i + i\frac{\alpha m f\psi_i\dot{r}_i}{2} + i\frac{\alpha m \dot{f}\psi_i r_i}{2} + i\frac{\alpha f\lambda\psi_0}{2r}\right) . \quad (14)$$

And, the super-charges Q and S are given by

$$Q = \frac{1}{\sqrt{m}}(m\psi_i\dot{r}_i - \psi_0\frac{\lambda}{r}) , \quad (15)$$

$$S = -tQ + \sqrt{m}\psi_i r_i . \quad (16)$$

There are also dynamical bosonic symmetries in the Lagrangian (4):

$$\begin{aligned} \delta_H r_i &= \dot{r}_i , & \delta_H \psi_\mu &= \dot{\psi}_\mu , & \text{time translation,} \\ \delta_D r_i &= t\dot{r}_i - \frac{1}{2}r_i , & \delta_D \psi_\mu &= t\dot{\psi}_\mu , & \text{dilation,} \\ \delta_K r_i &= t^2\dot{r}_i - tr_i , & \delta_K \psi_\mu &= t^2\dot{\psi}_\mu , & \text{conformal transformation.} \end{aligned} \quad (17)$$

The conserved bosonic charges corresponding to the bosonic symmetries are

$$\begin{aligned} H &= \frac{m}{2}\dot{r}_i^2 - \frac{e}{m}BS + \frac{\lambda^2}{2mr^2} - \frac{i\lambda}{mr^3}\psi_i\psi_0r_i , \\ D &= tH - \frac{1}{4}m(r_i\dot{r}_i + \dot{r}_i r_i) , \\ K &= -t^2H + 2tD + \frac{1}{2}mr^2 . \end{aligned} \quad (18)$$

The canonical momentum p_i of the Lagrangian (4) is given by

$$p_i = \frac{\partial L}{\partial \dot{r}_i} = m\dot{r}_i + eA_i , \quad (19)$$

and the dynamical quantum algebra is

$$\begin{aligned} [r_i, r_j] &= 0 , & [r_i, \dot{r}_i] &= i\delta_{ij}/m , \\ [\dot{r}_i, \dot{r}_j] &= i\epsilon_{ij}eB/m^2 , & \{\psi_\mu, \psi_\nu\} &= \delta_{\mu\nu} . \end{aligned} \quad (20)$$

Because the canonical anti-commutation relations for ψ_μ define a real Clifford algebra. ψ_μ can be given in terms of the Pauli matrices:

$$\psi_0 = \frac{\sigma_3}{\sqrt{2}} , \quad \psi_i = \epsilon_{ij} \frac{\sigma_j}{\sqrt{2}} , \quad (21)$$

so that $S = \frac{1}{2}\sigma_3$ as expected.

The conserved charges H, D, K, Q , and S satisfy the $Osp(1, 1)$ super-algebra on the plane except at the origin [8,9]:

$$\begin{aligned} [H, D] &= iH , & [H, K] &= 2iD , & [D, K] &= iK , \\ \{Q, Q\} &= 2H , & \{Q, S\} &= -2D , & \{S, S\} &= 2K , \\ [H, Q] &= 0 , & [K, S] &= 0 , & [H, S] &= -iQ , \\ [K, Q] &= iS , & [D, Q] &= -\frac{i}{2}Q , & [D, S] &= \frac{i}{2}S . \end{aligned} \quad (22)$$

Because the all charges H, D, K, Q , and S commute with J , full symmetry of the system is

$$G = SO(2)_{rotation} \otimes Osp(1, 1) . \quad (23)$$

The five charges are time-independent on the plane, because their total time derivatives vanish. For example, the total time derivative of the charge K is

$$\dot{K} = i[H, K] + \frac{\partial K}{\partial t} = 0 . \quad (24)$$

To construct all the states, we introduce the Cartan basis for the $Osp(1, 1)$ algebra

$$R = \frac{1}{2a^2}K + \frac{a^2}{2}H , \quad (25)$$

$$B_\pm = \frac{1}{2a^2}K - \frac{1}{2}a^2H \pm iD , \quad (26)$$

$$F_\pm = \frac{1}{2a}S \mp i\frac{a}{2}Q , \quad (27)$$

where a is a real positive parameter and has dimension $[t^{\frac{1}{2}}]$. R is a compact generator, and B_{\pm} and F_{\pm} are bosonic and fermionic raising/lowering operators respectively. In this Cartan basis, $Osp(1, 1)$ algebra becomes

$$\begin{aligned} [R, B_{\pm}] &= \pm B_{\pm} , \quad [R, F_{\pm}] = \pm \frac{1}{2} F_{\pm} , \\ [B_+, B_-] &= -2R , \quad [B_{\pm}, F_{\mp}] = \mp F_{\pm} , \\ \{F_+, F_-\} &= R , \quad \{F_{\pm}, F_{\pm}\} = B_{\pm} . \end{aligned} \quad (28)$$

The canonical chain for the group $Osp(1, 1)$ is

$$Osp(1, 1) \supset O(2, 1) \supset O(2) , \quad (29)$$

and the corresponding Casimirs of each groups are C , C_0 , and R . The Casimir C_0 of the subgroup $O(2, 1)$ is given by [10]

$$C_0 = \frac{1}{2}(HK + KH) - D^2 . \quad (30)$$

If we introduce the quantity $A = i[Q, S] - \frac{1}{2}$, then C_0 is

$$C_0 = \frac{1}{4}A^2 - \frac{A}{4} - \frac{3}{16} . \quad (31)$$

The Casimir of the super-group $Osp(1, 1)$ is

$$C = \frac{1}{2}\{H, K\} - D^2 + \frac{i}{4}[Q, S] + \frac{1}{16} = \frac{1}{4}A^2 , \quad (32)$$

which commutes with all the charges H , D , K , Q , and S [3]. In the coordinate representation, the operator A and C are given by

$$A = J\sigma_3 - \nu\sigma_3 - \frac{\lambda}{r}r_i\sigma_i , \quad (33)$$

$$C = \frac{1}{4}(J^2 + \nu^2 - 2J\nu + \lambda^2) . \quad (34)$$

We can choose a mutually commuting set of operators $\{C, C_0, R, J\}$, which can label the quantum states completely. However, C and C_0 are not independent, but can be expressed in terms of A from the Eqs. (31,32). Moreover, the magnitude of the eigenvalue of the operator A is not independent of the angular momentum. Therefore the eigenvalues of the following three operators can specify the quantum states completely:

$$\{J, \text{sign}(A), R\} . \quad (35)$$

III. SPECTRUM AND WAVE FUNCTIONS

We are ready to determine the spectrum and the wave functions using algebraic methods. To do so, we must know the eigenvalues of the three operators J , $sign(A)$, and R , which can label the quantum states. The eigenvalue of J is denoted by j . The eigenvalue of R is $(\Delta_{j,\alpha} + n)$, where $\Delta_{j,\alpha}$ is related to the eigenvalue of C_0 :

$$C_0 = \Delta_{j,\alpha}(\Delta_{j,\alpha} - 1) , \quad (36)$$

and $\alpha = \pm 1$ denotes the eigenvalue of $sign(A)$. Therefore, the expressions for the quantum numbers are

$$J | j, \alpha, n \rangle = j | j, \alpha, n \rangle , \quad (37)$$

$$A | j, \alpha, n \rangle = \alpha D_j | j, \alpha, n \rangle , \quad (38)$$

$$R | j, \alpha, n \rangle = (\Delta_{j,\alpha} + n) | j, \alpha, n \rangle , \quad (39)$$

where

$$D_j = \sqrt{j^2 + \nu^2 - 2j\nu + \lambda^2} , \quad (40)$$

$$\Delta_{j,\alpha} = \frac{1}{2} | D_j - \frac{\alpha}{2} | + \frac{1}{2} . \quad (41)$$

$\Delta_{j,\alpha}$ can be expressed in terms of D_j using the Eq. (31):

$$\Delta_{j,\alpha}(\Delta_{j,\alpha} - 1) = \frac{1}{4}D_j^2 - \frac{1}{4}D_j - \frac{3}{16} . \quad (42)$$

From the commutation relations (28), we obtain

$$B_{\pm} | j, \alpha, n \rangle = \sqrt{(\Delta_{j,\alpha} + n)(\Delta_{j,\alpha} + n \pm 1) - \Delta_{j,\alpha}(\Delta_{j,\alpha} - 1)} | j, \alpha, n \pm 1 \rangle , \quad (43)$$

$$F_{\pm} | j, \alpha, n \rangle = \sqrt{\frac{1}{2}(\Delta_{j,\alpha} + n) \pm \frac{1}{8} \pm \frac{1}{4}\alpha D_j} | j, -\alpha, n - \frac{1}{2}\alpha \pm \frac{1}{2} \rangle . \quad (44)$$

The lowest states can be obtained by the first order differential equation

$$B_- | j, \alpha, 0 \rangle = 0 . \quad (45)$$

From this equation, we have two lowest states $|j, 1, 0\rangle$ and $|j, -1, 0\rangle$. The higher states can be constructed by applying F_+ and B_+ on the two lowest states,

$$|j, \alpha, n\rangle = \sqrt{\frac{\Gamma(2\Delta_{j,\alpha})}{n! \Gamma(2\Delta_{j,\alpha} + n)}} B_+^n |j, \alpha, 0\rangle, \quad (46)$$

$$|j, -1, n\rangle = \frac{1}{\sqrt{\frac{1}{2}D_j + \frac{1}{4} + \frac{n}{2}}} F_+ |j, 1, n\rangle. \quad (47)$$

Because all the quantum states have been constructed, the corresponding wave functions can be easily obtained. The angular part of the wave function is found by obtaining the eigenvectors of the operator A . Assuming the normalized eigenvectors of A as

$$|j, \alpha\rangle = C_+^{(\alpha)} |j - \frac{1}{2}\rangle \otimes |\frac{1}{2}\rangle + C_-^{(\alpha)} |j + \frac{1}{2}\rangle \otimes |-\frac{1}{2}\rangle, \quad (48)$$

the angular eigenfunction can be given in the form

$$\begin{aligned} \eta_{j,\alpha}(\theta) &= \langle \theta | j, \alpha \rangle \\ &= \begin{pmatrix} C_+^{(\alpha)} e^{i(j-\frac{1}{2})\theta} \\ C_-^{(\alpha)} e^{i(j+\frac{1}{2})\theta} \end{pmatrix}. \end{aligned} \quad (49)$$

To determine $C_{\pm}^{(\alpha)}$, we use the Eq. (38) on the states

$$\begin{aligned} A \begin{pmatrix} C_+^{(\alpha)} e^{i(j-\frac{1}{2})\theta} \\ C_-^{(\alpha)} e^{i(j+\frac{1}{2})\theta} \end{pmatrix} &= \begin{pmatrix} j - \nu & -\lambda e^{-i\theta} \\ -\lambda e^{i\theta} & -j + \nu \end{pmatrix} \begin{pmatrix} C_+^{(\alpha)} e^{i(j-\frac{1}{2})\theta} \\ C_-^{(\alpha)} e^{i(j+\frac{1}{2})\theta} \end{pmatrix} \\ &= \alpha D_j \begin{pmatrix} C_+^{(\alpha)} e^{i(j-\frac{1}{2})\theta} \\ C_-^{(\alpha)} e^{i(j+\frac{1}{2})\theta} \end{pmatrix}. \end{aligned} \quad (50)$$

The ratio of $C_+^{(\alpha)}$ to $C_-^{(\alpha)}$ is

$$\frac{C_+^{(\alpha)}}{C_-^{(\alpha)}} = \frac{j - \nu + \alpha D_j}{-\lambda} = \frac{\lambda}{j - \nu - \alpha D_j}. \quad (51)$$

Therefore, the angular parts of the wave functions are

$$\eta_{j,\alpha}(\theta) = N_{j,\alpha} \begin{pmatrix} (j - \nu + \alpha D_j) e^{i(j-\frac{1}{2})\theta} \\ -\lambda e^{i(j+\frac{1}{2})\theta} \end{pmatrix}, \quad (52)$$

where $N_{j,\alpha}$ is the normalization factor.

The radial wave function $\Psi_{j,\alpha,n}(r)$ is constructed with the help of the identity;

$$R = \frac{1}{2a^2}K + \frac{a^2}{2K}(D^2 - iD + \frac{1}{4}A^2 - \frac{1}{4}A - \frac{3}{16}) . \quad (53)$$

The eigenvalue equation for R in the coordinate representation is

$$\langle r, \theta | R | j, \alpha, n \rangle = (\Delta_{j,\alpha} + n) \langle r, \theta | j, \alpha, n \rangle . \quad (54)$$

The full wave function can be written by

$$\langle r, \theta | j, \alpha, n \rangle = \Psi_{j,\alpha,n}(r) \eta_{j,\alpha}(\theta) . \quad (55)$$

Because the charges are time-independent, we can use the coordinate representations of K and D at $t = 0$:

$$K = \frac{1}{2}mr^2 , \quad D = \frac{i}{2}(r\frac{\partial}{\partial r} + 1) . \quad (56)$$

Therefore, the differential equation for $\Psi_{j,\alpha,n}(r)$ is

$$\frac{1}{2m}(-\frac{1}{r}\frac{d}{dr} - \frac{d^2}{dr^2} + \frac{(2\Delta_{j,\alpha} - 1)^2}{r^2} + \frac{m^2}{a^4}r^2)\Psi_{j,\alpha,n}(r) = \frac{2}{a^2}(\Delta_{j,\alpha} + n)\Psi_{j,\alpha,n}(r) . \quad (57)$$

We find the solutions of the above equation in terms of the generalized Laguerre polynomials,

$$\Psi_{j,\alpha,n}(r) = (-1)^n \left(\frac{m}{a^2}\right)^{\Delta_{j,\alpha}} \sqrt{\left(\frac{n!}{\pi\Gamma(2\Delta_{j,\alpha} + n)}\right)} r^{2\Delta_{j,\alpha}-1} \exp(-\frac{mr^2}{2a^2}) L_n^{2\Delta_{j,\alpha}-1}\left(\frac{mr^2}{a^2}\right) . \quad (58)$$

Of course, it is possible to obtain the above wave functions successively applying the coordinate representations of B_+ to the ground state wave function $\langle r, \theta | j, \alpha, 0 \rangle$.

Also, the energy eigenfunctions can be obtained using the following identity;

$$H = \frac{1}{K}(D^2 - iD + \frac{A^2}{4} - \frac{A}{4} - \frac{3}{16}) . \quad (59)$$

The radial Schrödinger equation is

$$\frac{1}{2m}(-\frac{1}{r}\frac{d}{dr} - \frac{d^2}{dr^2} + \frac{(2\Delta_{j,\alpha} - 1)^2}{r^2})\Psi_{j,\alpha}^E(r) = E\Psi_{j,\alpha}^E(r) . \quad (60)$$

The regular solutions to the above equation are

$$\Psi_{j,\alpha}^E(r) = \sqrt{m} J_{2\Delta_{j,\alpha}-1}(\sqrt{2mE} r) . \quad (61)$$

All the regular scattering wave functions vanish at the origin and respect the dynamical $Osp(1,1)$ symmetry. However, if $(2\Delta_{j,\alpha} - 1) < 1$, i.e.,

$$\nu - \sqrt{\frac{5}{4} + \alpha - \lambda^2} < j < \nu + \sqrt{\frac{5}{4} + \alpha - \lambda^2} , \quad (62)$$

then there exists one parameter family of square integrable self-adjoint singular wave functions which do not respect the dynamical symmetry at all:

$$\Psi_{j,\alpha}^E(r) = N(J_{2\Delta_{j,\alpha}-1}(\sqrt{2mE} r) + \xi J_{-2\Delta_{j,\alpha}+1}(\sqrt{2mE} r)) , \quad (63)$$

where N and ξ are the normalization and the real self-adjoint extension parameter respectively [11]. It is found that the number of these singular solutions is zero or two for an integer ν , one or three for a half integer ν , and at most three for a general ν from the Eq. (62).

IV. THE CASE WHEN AN EXTERNAL HARMONIC POTENTIAL IS ADDED

In this section, we show that the dynamical supersymmetry $Osp(1,1)$ of a spin $\frac{1}{2}$ charged particle interacting with a point vortex and $1/r^2$ potential can be imported to the case when an external harmonic oscillator potential is added.

We transform coordinates by

$$\begin{aligned} t &= \omega^{-1} \tan(\omega t') , \\ r_i &= r'_i \sec(\omega t') , \\ \psi_\mu &= \psi'_\mu . \end{aligned} \quad (64)$$

Then, the action for Lagrangian,

$$S = \int dt \left(\frac{1}{2} m \dot{r}_i^2 + e A_i \dot{r}_i - \frac{\lambda^2}{2mr^2} + \frac{i}{2} \psi_\mu \dot{\psi}_\mu + \frac{e}{m} B S + \frac{i\lambda}{mr^3} \psi_i \psi_0 r_i \right) , \quad (65)$$

is transformed into $S^h = \int dt' L^h$. The transformed \dot{r}_i , A_i , and B are

$$\begin{aligned} \dot{r}_i &= \dot{r}'_i \cos(\omega t') + \omega r'_i \sin(\omega t') , \quad A_i = A'_i \cos(\omega t') , \\ B &= B' \cos^2(\omega t') , \quad \dot{\psi}_\mu = \dot{\psi}'_\mu \cos^2(\omega t') . \end{aligned} \quad (66)$$

The new Lagrangian is

$$L^h = \frac{1}{2} m \dot{r}'_i{}^2 - \frac{1}{2} m \omega^2 r'^2 + e A'_i \dot{r}'_i - \frac{\lambda^2}{2 m r'^2} + \frac{i}{2} \dot{\psi}'_\mu \dot{\psi}'_\mu + \frac{e}{m} B' \mathcal{S}' + \frac{i\lambda}{m r'^3} \dot{\psi}'_i \dot{\psi}'_0 r'_i , \quad (67)$$

where $\dot{\psi}_i = \frac{d}{dt'} \psi$, $\dot{r}'_i = \frac{d}{dt'} r'_i$. And A'_i and B' are as the unchanged, but in terms of r'_i .

The corresponding new Hamiltonian is

$$\mathbf{H}^h = \frac{1}{2} m \dot{r}_i + \frac{1}{2} m \omega^2 r^2 + \frac{\lambda^2}{2 m r^2} - \frac{e}{m} B \mathcal{S} - \frac{i\lambda}{m r^3} \psi_i \psi_0 r_i , \quad (68)$$

where the primes are suppressed, and $m \dot{r}_i$ is $p_i - e A_i$. This Hamiltonian describes a spin $\frac{1}{2}$ particle interacting with a point vortex and a $1/r^2$ potential, but with the external harmonic oscillator potential added. Using the relation of the change of coordinate, we can obtain the imported charges

$$\begin{aligned} H^h &= \frac{1}{2} m \dot{r}_i^2 \cos^2(\omega t) + \frac{1}{4} m \omega (r_i \dot{r}_i + \dot{r}_i r_i) \sin(2\omega t) + \frac{1}{2} m \omega^2 r^2 \sin^2(\omega t) \\ &\quad - \frac{e}{m} B \mathcal{S} \cos^2(\omega t) + \cos^2(\omega t) \frac{\lambda^2}{2 m r^2} - \cos^2(\omega t) \frac{i\lambda}{m r^3} \psi_i \psi_0 r_i , \\ D^h &= \frac{1}{\omega} \tan(\omega t) H - \frac{1}{4} m (r_i \dot{r}_i + \dot{r}_i r_i) - \frac{1}{2} m \omega r^2 \tan(\omega t) , \\ K^h &= -\frac{1}{\omega^2} \tan^2(\omega t) H^h + \frac{2}{\omega} \tan(\omega t) D^h + \frac{1}{2} m r^2 \sec^2(\omega t) , \\ Q^h &= \sqrt{m} \cos(\omega t) \psi_i \dot{r}_i + \sqrt{m} \omega \sin(\omega t) \psi_i r_i - \cos(\omega t) \frac{\lambda}{\sqrt{m} r} \psi_0 , \\ S^h &= -\frac{1}{\omega} \tan(\omega t) Q^h + \sqrt{m} \sec(\omega t) \psi_i r_i . \end{aligned} \quad (69)$$

These charges are conserved with respect to the new Hamiltonian \mathbf{H}^h . From Eqs. (20,21), we can check they satisfy the supersymmetry $Sp(1,1)$.

Using the imported charges, we get at $t = 0$:

$$\begin{aligned} \mathbf{H}^h &= \omega^2 K^h + H^h = 2\omega R^h \big|_{a^2=\omega^{-1}} , \\ R^h &= \frac{1}{2a^2} K^h + \frac{a^2}{2} H^h . \end{aligned} \quad (70)$$

The eigenvalue equation for R^h is

$$R^h |j, \alpha, n\rangle = (\Delta_{j,\alpha} + n) |j, \alpha, n\rangle. \quad (71)$$

Thus, the energy spectrum of H^h is given by

$$E^h = 2(\Delta_{j,\alpha} + n)\omega. \quad (72)$$

The radial energy eigenfunctions are found using the identity for R^h ;

$$R^h = \frac{1}{2a^2}K^h + \frac{a^2}{2K^h}((D^h)^2 - iD^h + C_0). \quad (73)$$

Because the Casimir of C_0 is $\Delta_{j,\alpha}(\Delta_{j,\alpha} - 1)$, and the coordinate representation of D^h at $t = 0$ is $\frac{i}{2}(r\frac{\partial}{\partial r} + 1)$, the eigenvalue equation for R^h in the coordinate representation becomes

$$\frac{1}{2m}\left(-\frac{1}{r}\frac{d}{dr} - \frac{d^2}{dr^2} + \frac{(2\Delta_{j,\alpha} - 1)^2}{r^2} + m^2\omega^2 r^2\right)\Psi_{j,\alpha,n}^h(r) = 2\omega(\Delta_{j,\alpha} + n)\Psi_{j,\alpha,n}^h(r). \quad (74)$$

Therefore, the radial wavefunctions are

$$\Psi_{j,\alpha,n}^h(r) = (-1)^n(m\omega)^{\Delta_{j,\alpha}} \sqrt{\left(\frac{n!}{\pi\Gamma(2\Delta_{j,\alpha} + n)}\right)} r^{2\Delta_{j,\alpha}-1} \exp\left(-\frac{1}{2}m\omega r^2\right) L_n^{2\Delta_{j,\alpha}-1}(m\omega r^2). \quad (75)$$

The angular part of the wavefunction is the same as Eq. (52).

V. CONCLUSION

We have shown that the quantum mechanical system of a spin 1/2 charged particle interacting with a point magnetic vortex and $1/r^2$ potential possesses the dynamical symmetry $Osp(1,1)$. Using this dynamical symmetry, the spectrum and wave functions of the system could be constructed. And, the dynamical supersymmetry could be imported to the case when additional harmonic potential is present.

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REFERENCES

- [1] R. Jackiw, Ann. Phys. (NY) **129**, 183 (1980).
- [2] E. D'Hoker and L. Vinet, Phys. Lett. **137B**, 72 (1984).
- [3] E. D'Hoker and L. Vinet, Comm. Math. Phys. **97**, 391 (1985).
- [4] R. Jackiw, Ann. Phys. (NY) **201**, 83 (1990).
- [5] C.J. Park, Nucl. Phys. B **376**, 99 (1992).
- [6] J.G. Demers, Mod. Phys. Lett. **8A**, 827 (1993).
- [7] F.A. Berezin and M.S. Marinov, J.E.T.P. Lett. **21**, 30 (1975), Ann. Phys. (NY) **104**, 336 (1977).
- [8] A. Paris and V. Rittenberg, J. of Math. Phys. **16**, 2062(1975).
- [9] M. Scheunert, W. Nahm and V. Rittenberg, J. of Math. Phys. **18**, 155 (1977).
- [10] B. Wybourne, *Classical Groups for Physicist*, (New York: Wiley 1974).
- [11] R. Jackiw, "M.A.B. Bég Memorial Volume" (A. Ali and P. Hoodbhoy, Eds.), World Scientific, Singapore.